Math 150, ***Lecture Notes- Bonds*** **Name____________**

Section 2.6 *Related Rates*

Finding Related Rates

You have seen how the Chain Rule can be used to find dy/dx implicitly. Another important use of the Chain Rule is to find the rates of change of two or more related variables that are changing with respect to *time*.

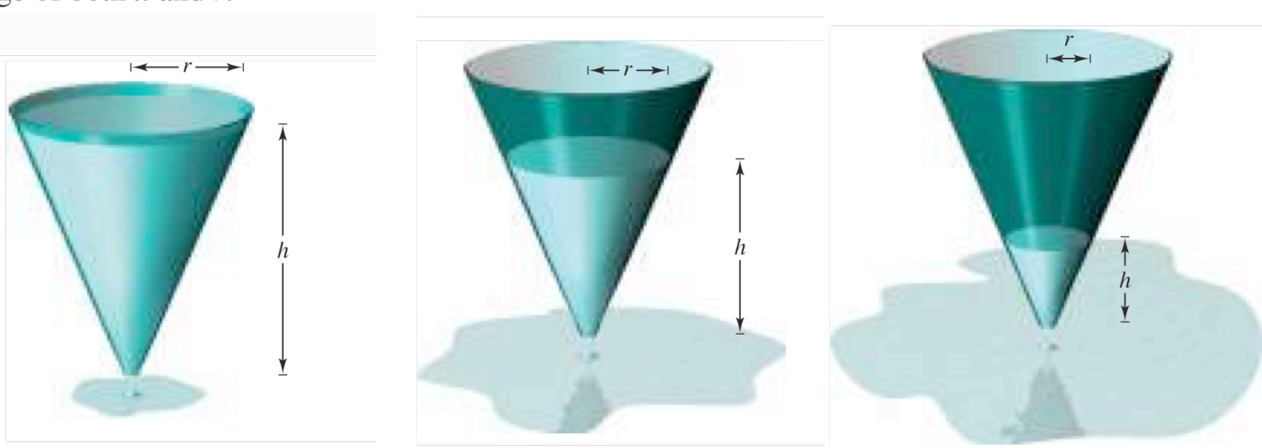
For example, when water is drained out of a conical tank (see Figure 2.33), the volume V, the radius r, and the height h of the water level are all functions of time t. Knowing that these variables are related by the equation

$$V = \frac{\pi}{3} r^2 h \qquad \text{Original equation}$$

you can differentiate implicitly with respect to t to obtain the **related-rate** equation

$$\begin{aligned} \frac{d}{dt}(V) &= \frac{d}{dt}\left(\frac{\pi}{3} r^2 h\right) \\ \frac{dV}{dt} &= \frac{\pi}{3}\left[r^2 \frac{dh}{dt} + h\left(2r\frac{dr}{dt}\right)\right] \qquad \text{Differentiate with respect to } t. \\ &= \frac{\pi}{3}\left(r^2 \frac{dh}{dt} + 2rh\frac{dr}{dt}\right). \end{aligned}$$

From this equation you can see that the rate of change of V is related to the rates of change of both h and r.



EXPLORATION

Finding a Related Rate In the conical tank shown in Figure 2.33, suppose that the height of the water level is changing at a rate of -0.2 foot per minute and the radius is changing at a rate of -0.1 foot per minute. What is the rate of change in the volume when the radius is $r = 1$ foot and the height is $h = 2$ feet? Does the rate of change in the volume depend on the values of r and h? Explain.

Guidelines For Solving Related-Rate Problems

1. Identify all *given* quantities and quantities *to be determined*. Make a sketch and label the quantities.
2. Write an equation involving the variables whose rates of change either are given or are to be determined.
3. Using the Chain Rule, implicitly differentiate both sides of the equation *with respect to time t*.
4. *After* completing Step 3, substitute into the resulting equation all known values for the variables and their rates of change. Then solve for the required rate of change.

The table below lists examples of mathematical models involving rates of change. For instance, the rate of change in the first example is the velocity of a car.

Verbal Statement	Mathematical Model
The velocity of a car after traveling for 1 hour is 50 miles per hour.	$x =$ distance traveled $\frac{dx}{dt} = 50$ when $t = 1$
Water is being pumped into a swimming pool at a rate of 10 cubic meters per hour.	$V =$ volume of water in pool $\frac{dV}{dt} = 10\ \text{m}^3/\text{hr}$
A gear is revolving at a rate of 25 revolutions per minute (1 revolution $= 2\pi$ rad).	$\theta =$ angle of revolution $\frac{d\theta}{dt} = 25(2\pi)$ rad/min

$x(t)$ $y(t)$

Ex.1 Assume that x and y are differentiable functions of t. Find the required values of $\frac{dx}{dt}$ and $\frac{dy}{dt}$. Given $x^2 + y^2 = 25$,

$$\frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}(25)$$

(a) find $\frac{dy}{dt}$ when $\frac{dx}{dt} = 8$, $x = 3$ and $y = 4$.

$$\frac{d}{dt}(x^2) + \frac{d}{dt}(y^2) = 0$$

$$2x \cdot \frac{dx}{dt} + 2y \cdot \frac{dy}{dt} = 0$$

$$\boxed{\frac{dy}{dt} = ?}$$

$$2(3)\cdot(8) + 2(4)\frac{dy}{dt} = 0$$

$$48 + 8\frac{dy}{dt} = 0$$

$$8\frac{dy}{dt} = -48$$

$$\frac{1}{8} \cdot 8\frac{dy}{dt} = (-48)\cdot\frac{1}{8}$$

$$\boxed{\frac{dy}{dt} = -6}$$

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$ $\qquad x^2+y^2=25$

Ex.1

(b) find $\dfrac{dx}{dt}$ when $\dfrac{dy}{dt}=-2$, $x=4$ and $y=3$.

$$2(4)\cdot\frac{dx}{dt} + 2(3)\cdot(-2)=0$$

$$8\cdot\frac{dx}{dt} - 12 = 0$$

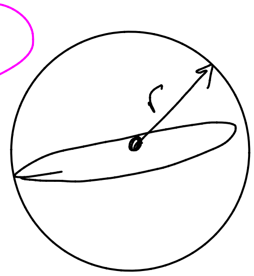
$$8\frac{dx}{dt} = 12$$

$$\frac{1}{8}\cdot 8\frac{dx}{dt} = \frac{1}{8}\cdot\frac{12}{1}$$

$$\boxed{\frac{dx}{dt} = \frac{3}{2}}$$

r = radius of the sphere in inches

D



R

$\dfrac{dr}{dt} = +3\,\dfrac{in}{min}$

Ex.2

Volume The radius r of a sphere is increasing at a rate of 3 inches per minute.

$\dfrac{dV}{dt} = \quad \dfrac{in^3}{min}$ + ← Bigger

(a) Find the rates of change of the volume when $r = 9$ inches and $r = 36$ inches.

(b) Explain why the rate of change of the volume of the sphere is not constant even though dr/dt is constant.

E

Equation: $\qquad V = \dfrac{4}{3}\pi r^3$

D

$$\frac{d}{dt}(V) = \frac{d}{dt}\left[\frac{4}{3}\pi r^3\right]$$

$$\frac{dV}{dt} = \frac{4}{3}\pi\cdot 3r^2\cdot\frac{dr}{dt}$$

$$\boxed{\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}}$$

S. r = 9 inches

$$\frac{dV}{dt} = 4\pi(9in)^2\cdot\left(\frac{3in}{min}\right)$$

$$= 4\pi(81)3\,\frac{in^3}{min}$$

$$\frac{dV}{dt} = 972\pi\,\frac{in^3}{min}$$

$$\frac{dV}{dt} = 4\pi(36in)^2\cdot\left(\frac{3in}{min}\right)$$

$$\frac{dV}{dt} = 4\pi\cdot 1296\cdot 3\,\frac{in^3}{min}$$

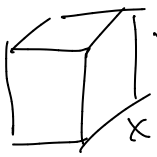
$$\frac{dV}{dt} = 15{,}552\pi\,\frac{in^3}{min}$$

t = time in second

D
R
E
D

Ex.3 "cube"

Volume All edges of a cube are expanding at a rate of 6 centimeters per second. How fast is the volume changing when each edge is (a) 2 centimeters and (b) 10 centimeters?



x = height in cm

x = width in cm

x = length in cm

$$\frac{dV}{dt} = + \quad \frac{cm^3}{sec}$$

↑ "bigger"

$$\frac{dx}{dt} = +6 \frac{cm}{sec}$$

V = Volume of a Cube

$V = (x)(x)(x)$

E. $V = x^3$

$$V = x^3$$

$$\frac{d}{dt}(V) = \frac{d}{dt}(x^3)$$

$$\frac{dV}{dt} = 3x^2 \cdot \frac{dx}{dt}$$

(a) If $x = 2cm$:

$$\frac{dV}{dt} = 3(2cm)^2 \cdot \left(\frac{6 cm}{sec}\right)$$

$$\frac{dV}{dt} = 3 \cdot 4 \cdot 6 \frac{cm^3}{sec}$$

$$\boxed{\frac{dV}{dt} = 72 \frac{cm^3}{sec}}$$

(a) The Volume is increasing at a rate of 72 cm^2/sec when the sides are 2cm.

(b) If $x = 10 cm$

$$\frac{dV}{dt} = 3(10cm)^2 \cdot \left(\frac{6 cm}{sec}\right)$$

$$\frac{dV}{dt} = 3 \cdot 100 \cdot 6 \frac{cm^3}{sec}$$

$$\frac{dV}{dt} = 1{,}800 \frac{cm^3}{sec}$$

(b) The Volume is increasing at a rate of 1,800 cm^3/sec when the sides are 10 cm.

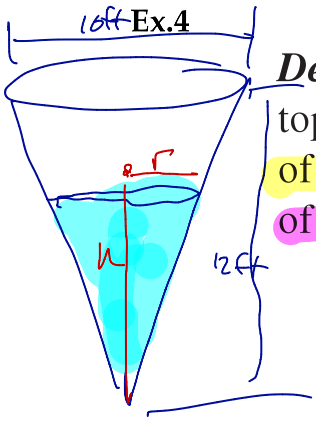
t = time in minutes

r = radius of the surface of the water

h = depth of the water

V = Volume of the water in a cone

Ex.4



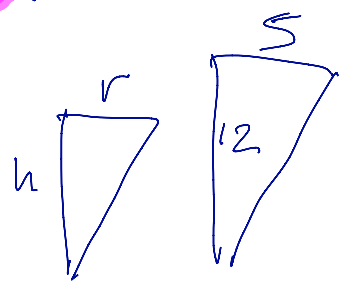
Depth A conical tank (with vertex down) is 10 feet across the top and 12 feet deep. If water is flowing into the tank at a rate of 10 cubic feet per minute, find the rate of change of the depth of the water when the water is 8 feet deep.

$$\frac{dV}{dt} = +\,10\,\frac{ft^3}{min}$$

$$\frac{dh}{dt} = +\ \frac{ft}{min}$$

D.
R.
E.
D

$$V = \frac{\pi}{3} r^2 h$$



$$\frac{h}{r} = \frac{12}{5}$$

$$5h = 12r$$

$$\frac{5h}{12} = r$$

$$V = \frac{\pi}{3}\left(\frac{5h}{12}\right)^2 h$$

$$\boxed{V = \frac{25\pi}{432} h^3}$$

$$\frac{d}{dt}(V) = \frac{d}{dt}\left[\frac{25\pi}{432} h^3\right]$$

$$\frac{dV}{dt} = \frac{25\pi}{432} \cdot 3h^2 \frac{dh}{dt}$$

$$\boxed{\frac{dV}{dt} = \frac{25\pi}{144} h^2 \frac{dh}{dt}}$$

$$10\,\frac{ft^3}{min} = \frac{25\pi}{144} (8\,ft)^2 \frac{dh}{dt}$$

$$10\,\frac{ft^3}{min} = \frac{25\pi}{144} \cdot 64\,ft^2 \cdot \frac{dh}{dt}$$

$$10\,\frac{ft^3}{min} = \frac{4}{9} \cdot 25\pi \cdot ft^2 \cdot \frac{dh}{dt}$$

$$10\,\frac{ft^3}{min} = \frac{100\pi}{9} ft^2 \frac{dh}{dt}$$

$$\frac{9}{100\pi\,ft^2}\left(\frac{10\,ft^3}{min}\right) = \frac{dh}{dt}$$

$$\frac{9}{10\pi}\,\frac{ft}{min} = \frac{dh}{dt}$$

$$0.29\,\frac{ft}{min} \approx \frac{dh}{dt}$$

The depth of the water is increasing at a rate of approximately 0.29 ft/min.

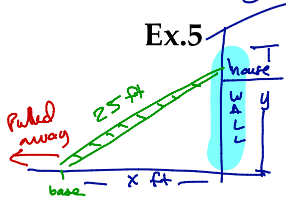
On the Test

Let x = distance – wall to the base, (ft)
t = time in seconds
y = height of the ladder along the wall (ft)

$$\frac{dx}{dt} = +2\,\frac{ft}{sec}$$

$$\frac{dy}{dt} = ?\ \frac{ft}{sec}$$

Ex.5



Moving Ladder A ladder 25 feet long is leaning against the wall of a house (see figure). The base of the ladder is pulled away from the wall at a rate of 2 feet per second.

(a) How fast is the top of the ladder moving down the wall when its base is 7 feet, 15 feet, and 24 feet from the wall?

(b) Consider the triangle formed by the side of the house, the ladder, and the ground. Find the rate at which the area of the triangle is changing when the base of the ladder is 7 feet from the wall.

(c) Find the rate at which the angle between the ladder and the wall of the house is changing when the base of the ladder is 7 feet from the wall.

D
R
E
D

$7^2 + ?^2 = 25^2$
$49 + ?^2 = 625$
$?^2 = 576$
$? = 24$

$15^2 + w^2 = 25^2$
$225 + w^2 = 625$
$w^2 = 400$
$w = 20$

(a) Equ $\quad x^2 + y^2 = 25^2$

$$\frac{d}{dt}[x^2 + y^2] = \frac{d}{dt}[25^2]$$

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

(a)

$$2(7\,ft)\left(\frac{2\,ft}{sec}\right) + 2(24\,ft)\left(\frac{dy}{dt}\right) = 0$$

$$28\,\frac{ft^2}{sec} + 48\,ft\cdot\frac{dy}{dt} = 0$$

$$\frac{1}{48\,ft}\left[48\,ft\cdot\frac{dy}{dt}\right] = -28\,\frac{ft^2}{sec}\left[\frac{1}{48\,ft}\right]$$

$$\boxed{\frac{28}{48} = \frac{2\cdot2\cdot7}{2\cdot2\cdot12}}$$

$$\frac{dy}{dt} = -\frac{7}{12}\,\frac{ft}{sec}$$

(b)

$$2(15\,ft)\left(\frac{2\,ft}{sec}\right) + 2(20\,ft)\frac{dy}{dt} = 0$$

$$60\,\frac{ft^2}{sec} + 40\,ft\cdot\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{60}{40}\,\frac{ft}{sec}$$

$$\frac{dy}{dt} = -\frac{3}{2}\,\frac{ft}{sec}$$

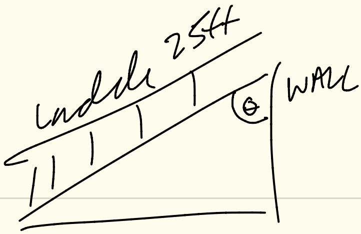
(c)

$$2(24\,ft)\left(2\,\frac{ft}{sec}\right) + 2(7\,ft)\frac{dy}{dt} = 0$$

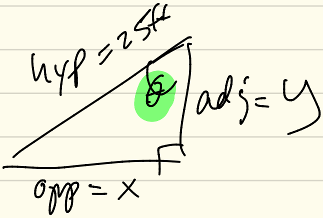
$$96\,\frac{ft^2}{sec} + 14\,ft\,\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{96}{14}\,\frac{ft}{sec}$$

$$\frac{dy}{dt} = -\frac{48}{7}\,\frac{ft}{sec}$$



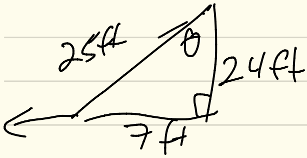
Let θ = angle between ladder & wall



$$\frac{d\theta}{dt} = + \quad \frac{\text{RAD}}{\text{sec}}$$

$$\frac{dy}{dt} = -\frac{7}{12} \ \frac{\text{ft}}{\text{sec}}$$

$$\tan(\theta) = \frac{\text{opp}}{\text{adj}} = \frac{x}{y}$$



$$\sin(\theta) = \frac{x}{25}$$

$$\frac{dx}{dt} = +2 \ \frac{\text{ft}}{\text{sec}}$$

$$\frac{d}{dt}[\sin(\theta)] = \frac{d}{dt}\left(\frac{x}{25}\right)$$

$$\cos(\theta) = \frac{\text{ADJ}}{\text{Hyp}} = \frac{24}{25}$$

$$\cos(\theta) \cdot \frac{d\theta}{dt} = \frac{1}{25} \cdot \frac{dx}{dt}$$

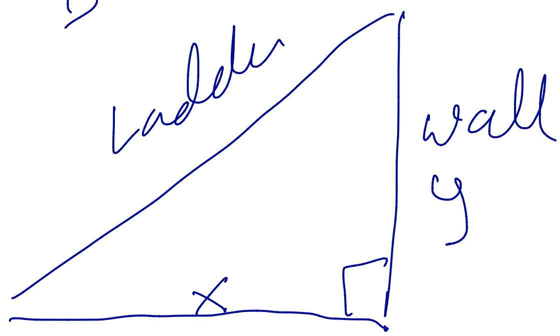
$$\frac{d\theta}{dt} = \frac{1}{\cos(\theta)} \cdot \frac{1}{25} \cdot \frac{dx}{dt}$$

$$\frac{d\theta}{dt} = \frac{25}{24} \cdot \frac{1}{25\text{ft}}\left(2\ \frac{\text{ft}}{\text{sec}}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{12} \ \frac{\text{RAD}}{\text{sec}}$$

Ex.5 – cont'd

D



Let

R

A = Area of the triangle

$\frac{dA}{dt} = \downarrow \quad \frac{ft^2}{sec}$

E

Area = $\frac{1}{2}$ · base · height

Eqn $\quad A = \frac{1}{2} \cdot x \cdot y$

D $\quad \frac{d}{dt}(A) = \frac{1}{2} \cdot \frac{d}{dt}[x \cdot y]$

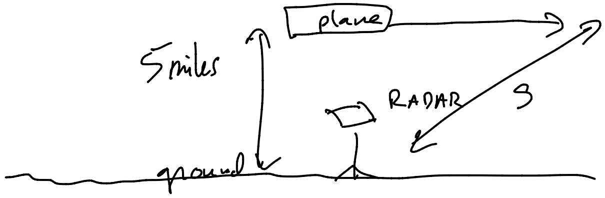
$$\frac{dA}{dt} = \frac{1}{2}\left[x \cdot \frac{dy}{dt} + y \cdot \frac{dx}{dt}\right]$$

S $\quad \frac{dA}{dt} = \frac{1}{2}\left[(7ft)\left(-\frac{7}{12}\frac{ft}{sec}\right) + (24\ ft)\left(2\ \frac{ft}{sec}\right)\right]$

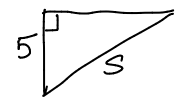
$$\frac{dA}{dt} = \frac{1}{2}\left[-\frac{49}{12}\ \frac{ft^2}{sec} + 48\ \frac{ft^2}{sec}\right]$$

$$\frac{dA}{dt} = \frac{527}{24}\ \frac{ft^2}{sec}$$

D



y = distance plane travels



D
R
E
D
S

Ex.6

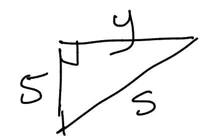
let s = distance between plane & RADAR STATION

Air Traffic Control An airplane is flying at an altitude of 5 miles and passes directly over a radar antenna (see figure). When the plane is 10 miles away ($s = 10$), the radar detects that the distance s is changing at a rate of 240 miles per hour. What is the speed of the plane?

$\left.\dfrac{ds}{dt}\right|^{+} = \dfrac{240 \text{ miles}}{1 \text{ hour}}$

$s = 10$ miles

Eqn: $5^2 + y^2 = s^2$



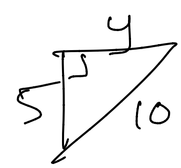
D:

$$\frac{d}{dt}[25 + y^2] = \frac{d}{dt}[s^2]$$

$\dfrac{dy}{dt}^{+} = \dfrac{\text{miles}}{\text{hour}}$

$$\frac{d}{dt}(25) + \frac{d}{dt}(y^2) = 2s\cdot\frac{ds}{dt}$$

$$0 + 2y\cdot\frac{dy}{dt} = 2s\frac{ds}{dt}$$



$$\boxed{y\frac{dy}{dt} = s\frac{ds}{dt}}$$

$5^2 + y^2 = 10^2$

$25 + y^2 = 100$

$y^2 = 75$

$y = \sqrt{25}\sqrt{3}$

$y = 5\sqrt{3}$

S:

$$(5\sqrt{3} \text{ miles})\cdot\frac{dy}{dt} = (10 \text{ miles})\left(\frac{240 \text{ miles}}{\text{hour}}\right)$$

$$\frac{dy}{dt} = \left(\frac{1}{\cancel{5}\sqrt{3} \text{ miles}}\right)(\cancel{10}^{\,2} \text{ miles})\left(\frac{240 \text{ miles}}{\text{hour}}\right)$$

$$\frac{dy}{dt} = \frac{480}{\sqrt{3}} \frac{\text{miles}}{\text{hour}} = \frac{160\cdot\cancel{\sqrt{3}}^{\,3}\cdot\sqrt{3}}{\cancel{\sqrt{3}}} \text{ miles}$$

$$\frac{dy}{dt} = 160\sqrt{3}\ \frac{\text{miles}}{\text{hour}}$$

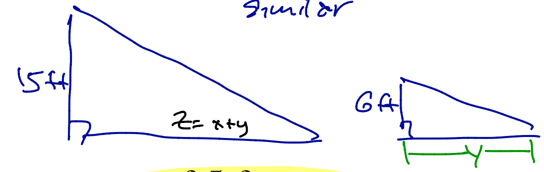
$$\frac{dy}{dt} \approx 277 \text{ mph}$$

The plane is traveling approximately 277 miles per hour.

t = time in second

tip of the shadow

"similar"



Ex.7

$\frac{dx}{dt} = +5\frac{ft}{sec}$

Shadow Length A man 6 feet tall walks at a rate of 5 feet per second away from a light that is 15 feet above the ground (see figure). When he is 10 feet from the base of the light,

x = distance from light post to the man

(a) at what rate is the tip of his shadow moving?

(b) at what rate is the length of his shadow changing?

y = length of the shadow

$\frac{dy}{dt} = + \frac{ft}{sec}$

z = distance from the pole to the tip of the shadow

$\frac{dz}{dt} = + \frac{ft}{sec}$

D
R
E
D
S

Eqn

$$\frac{15}{x+y} = \frac{6}{y}$$

$$15y = 6(x+y)$$

$$15y = 6x + 6y$$

$$\boxed{9y = 6x}$$

$$\frac{d}{dt}[9y] = \frac{d}{dt}[6x]$$

$$\frac{9 \cdot \frac{dy}{dt}}{9} = \frac{6 \cdot \frac{dx}{dt}}{9}$$

S

$$\frac{dy}{dt} = \frac{2}{3}\frac{dx}{dt}$$

$$\frac{dy}{dt} = \frac{2}{3}\left(\frac{5ft}{sec}\right)$$

(b)

$$\boxed{\frac{dy}{dt} = \frac{10}{3}\frac{ft}{sec}}$$

(a) Eqn

$$\frac{15}{z} = \frac{6}{y}$$

$$15y = 6z$$

$$\frac{d}{dt}[15y] = \frac{d}{dt}[6z]$$

$$\frac{15 \cdot \frac{dy}{dt}}{6} = \frac{6 \cdot \frac{dz}{dt}}{6}$$

S

$$\frac{5}{2} \cdot \frac{dy}{dt} = \frac{dz}{dt}$$

$$\frac{5}{2} \cdot \left(\frac{10\,ft}{3\,sec}\right) = \frac{dz}{dt}$$

$$\boxed{\frac{25}{3}\frac{ft}{sec} = \frac{dz}{dt}}$$

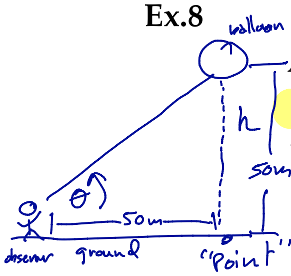
let h = height of the balloon in meters | $\frac{dh}{dt} = +\frac{4\text{m}}{\text{sec}}$

θ = angle of elevation RAD $\quad \frac{d\theta}{dt} = + \frac{\text{RAD}}{\text{Sec}}$

t = time in seconds

D R E D S

Ex.8



Angle of Elevation A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.

Eqn:

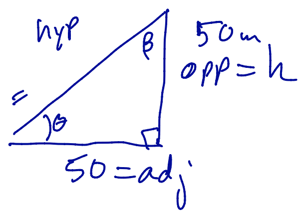
$$\tan(\theta) = \frac{h}{50}$$

$$\frac{d}{dt}[\tan(\theta)] = \frac{1}{50}\cdot\frac{d}{dt}(h)$$

$$\sec^2(\theta)\cdot\frac{d\theta}{dt} = \frac{1}{50}\cdot\frac{dh}{dt}$$

$$\boxed{\cos^2(\theta)\cdot\sec^2(\theta)}\cdot\frac{d\theta}{dt} = \cos^2(\theta)\cdot\frac{1}{50}\cdot\frac{dh}{dt}$$

$$\boxed{\frac{d\theta}{dt} = \frac{\cos^2(\theta)}{50}\cdot\frac{dh}{dt}}$$



$$\tan(\theta) = \frac{\text{opp}}{\text{adj}}$$

$$\tan(\theta) = \frac{h}{50}$$

$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{50}{50\sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

$$50^2 + 50^2 = \text{hyp}^2$$

$$\sqrt{2\cdot 50^2} = \text{hyp}$$

$$50\sqrt{2} = \text{hyp}$$

$$\frac{d\theta}{dt} = \frac{\left(\frac{\sqrt{2}}{2}\right)^2}{50\text{ m}}\left(\frac{+4\text{ m}}{\text{sec}}\right)$$

$$\frac{d\theta}{dt} = \frac{\frac{2}{4}\cdot 4}{50}\ \frac{\text{RAD}}{\text{sec}}$$

$$\frac{d\theta}{dt} = \frac{1}{25}\ \frac{\text{RAD}}{\text{sec}}$$

ON the Test

Ex.8

Angle of Elevation A balloon rises at a rate of 4 meters per second from a point on the ground 50 meters from an observer. Find the rate of change of the angle of elevation of the balloon from the observer when the balloon is 50 meters above the ground.